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# COVERS COUNTING VIA FEYNMAN CALCULUS

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*Let $G$ be a finite group. In this paper, we present a tool for counting the number of principal $G$-bundles over a surface. As an application, we express (nonstandard) generating functions for the double Hurwitz numbers as integrals over commutative Frobenius algebras associated with symmetric groups. Bibliography:* 11 *titles.*

## 1. Introduction

This note is devoted to a special case of the following big problem.

*Big problem:* Let $M = \{\mu_1, \dots, \mu_k\}$ be a finite ordered collection of conjugacy classes of a finite group $G$. Count the weighted number of principal $G$-bundles over a closed oriented surface $\Sigma$ of genus $g$ with $k$ marked points $q_1, \dots, q_k$ such that the holonomy around $q_i$ belongs to the class $\mu_i$ for $i = 1, \dots, k$.

Here "weighted" means that we count each bundle with the weight reciprocal to the number of its automorphisms.

The significance of this problem is the computation of correlation functions of a certain two-dimensional topological quantum field theory.

An efficient way to deal with it is to use its connection with the combinatorics and representation theory of the group $G$. Namely, if $\#N(M)$ is the number of homomorphisms from the fundamental group of $\Sigma/\{q_1, \dots, q_k\}$ to $G$ such that the image of the element corresponding to the complete revolution around $q_i$ is contained in the conjugacy class corresponding to $\mu_i$, then the number of bundles is the ratio of $\#N(M)$ to the order of $G$. For the details, see [9].

We use this connection for counting the number of bundles of a special kind. Namely, denote by $h_\tau(\mu, \nu; r)$ the weighted number of principal $G$-bundles over a 2-sphere with a nontrivial holonomy around $r + 2$ points $q_0, \dots, q_{r+1}$ such that the holonomy around $q_0$ is in the conjugacy class $\mu$, the holonomy around $q_{r+1}$ is in the conjugacy class $\nu$, and the holonomy around all the other points belongs to a fixed conjugacy class $\tau$.

The main result of this note is the following. We construct generating functions for the numbers $h_\tau(\mu, \nu; r)$ as integrals over the center of the group algebra $\mathbb{C}[G]$. Namely, for an explicitly constructable square matrix $A_\tau(\beta)$, we have

$$\sum_{k=0}^{\infty} h_\tau(\mu, \nu; r)\beta^r = \frac{1}{Z_\tau} \frac{\mathrm{tr}(f_{\mu^{-1}} f_\mu)\mathrm{tr}(f_{\nu^{-1}} f_\nu)}{\#G} \int\limits_{Z\mathbb{C}[G]} z_\mu \bar{z}_\nu e^{-\langle A_\tau(\beta) z, z\rangle} \, dm,$$

where all notation is explained in the text. This presentation allows us to express the generating functions for the numbers $h_\tau(\mu, \nu; r)$ as entries of the matrix $(A_\tau(\beta))^{-1}$ multiplied by certain constants.

As a corollary of this presentation, we find that the corresponding generating functions are rational functions in $\beta$. We also derive a nonlinear differental equation for these functions.

The main example for our considerations is given by the (disconnected) double Hurwitz numbers. This is the case where the group $G$ is a symmetric group $S_d$ and the distinguished class $\tau$ is the class of a transposition. Principal $S_d$-bundles are just ramified covers of degree $d$. These numbers have a very rich structure related to various fields of mathematics.

In the case where one of the holonomies (say, $\nu$) is in the conjugacy class of the unit, the celebrated ELSV formula (see, for example, the original paper [4]) relates the numbers of connected covers with intersection numbers on the moduli space of complex curves.

The geometric meaning of the double Hurwitz numbers is not yet completely understood. In the paper [7], the authors conjectured that if $\nu = (d)$, then there exists a moduli space $\mathrm{Pic}_{g,n}$ such that these numbers can be expressed in terms of intersection numbers on some properly chosen compactification. They have also found an expression for generating functions for the double Hurwitz numbers in terms of Schur polynomials, and found an explicit formula for the one-part double Hurwitz numbers (this is the case where there is a complete branching

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over one of the special points). In the paper [11], it was shown that a generating function for the double Hurwitz numbers is a $\tau$-function for the Toda lattice hierarchy.

Later, in [2] it was found that the computation of the double Hurwitz numbers can be carried out in the language of tropical geometry. In [1], the authors give a method for computing the number of covers with an arbitrary ramification. In a sense, our work is a continuation of the research carried out in [1].

The paper [6] provides the following formula for a generating function for the double Hurwitz numbers, generalizing the results of [7]. Let the cyclic type of $\mu$ be $(\mu_i)_{i=1}^{l(\mu)}$ and the cyclic type of $\nu$ be $(\nu_j)_{j=1}^{l(\nu)}$. By $\varsigma(z)$ we denote the following function:

$$\varsigma(z) = e^{z/2} - e^{-z/2}.$$

Then, for $\mu$ and $\nu$ satisfying a certain condition (the pair $(\mu,\, \nu)$ is contained in a *chamber* of $R_{l(\mu),l(\nu)}$; for the details, see [6]), we have

$$\sum_{r=0}^{\infty} h_\tau(\mu,\nu;r)\frac{z^r}{r!} = \frac{1}{(\#\mathrm{Aut}\ \mu)\prod \mu_i}\frac{1}{(\#\mathrm{Aut}\ \nu)\prod \nu_j}\frac{1}{\varsigma(dz)}\sum_{k=1}^{t(\mu,\nu)}\prod_{m=1}^{l(\mu)+l(\nu)+1}\varsigma(zQ_{k,l}^{\mu,\nu}),$$

where $t(\mu,\nu)$ is a finite number and $Q_{k,l}^{\mu,\nu}$ are certain quadratic polynomials in $\mu_i$ and $\nu_j$.

This result is obtained using the infinite wedge space formalism.

In the language of the infinite wedge space, the application of our technique to the problem of computing the double Hurwitz numbers is rather trivial. Namely, we study the vacuum expectations of operators of the form

$$\left\langle \prod \alpha_{\mu_i}(1-\beta\mathcal{F}_2)^{-1}\prod \alpha_{-\nu_j}\right\rangle,$$

which is nothing more but the study of the entries of the block-diagonal operator $(1-\beta\mathcal{F}_2)^{-1}$ in the basis $\prod \alpha_{-\nu_j}|0\rangle$.

Our technique allows us to produce generating functions for the double Hurwitz numbers even in the case where the pair $(\mu,\nu)$ belongs to a *resonance arrangement* in the terminology of [6]. It works also in the case of an arbitrary finite group.

The paper is organized as follows. A presentation of the fundamental group of a surface is constructed in Sec. 2. Section 3 presents a reformulation of the enumeration problem in the algebraic language. Section 4 is a short overview of Feynman calculus, and Sec. 5 is devoted to the application of this technique to our problem. Section 6 contains the calculation of the (disconnected) double Hurwitz numbers for the degrees $d = 2,3,4$. We also give a comparison of our generating function with that of [6] in one particular case.

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## 2. The group presentation corresponding to a pair-of-pants decomposition of a surface

We begin with a special presentation of the fundamental group of a punctured surface.

**Definition 2.1.** *Let $g$ and $n$ be two nonnegative integers such that $2-2g-n < 0$. Denote by $H_{g,n}$ the fundamental group of a connected oriented surface of genus $g$ with $n$ punctures.*

Let us define some notions we are going to use for constructing a presentation of $H_{g,n}$.

**Definition 2.2.** *By a* 1-3-valent graph $\Gamma$ *we mean a finite graph having only vertices of valence 1 and 3. The first Betti number of $\Gamma$ (regarded as a one-dimensional cell complex) is referred to as its* genus. *The 1-valent vertices of $\Gamma$ are referred to as the* ends; *edges incident to ends are called* leaves; *all the other edges are called* inner edges. *The set of all vertices of $\Gamma$ that are not ends is called the* set of inner vertices *and is denoted by $V^0(\Gamma)$. The set of all edges of $\Gamma$ is denoted by $E(\Gamma)$; the set of inner edges of $\Gamma$ is denoted by $E^0(\Gamma)$. The set of edges of $\Gamma$ adjacent to $v \in V^0(\Gamma)$ is denoted by $E_v$.*

From now on, all the graphs we are dealing with are supposed to be 1-3-valent unless otherwise stated.

Another notion we are going to use is the following.

**Definition 2.3.** *Let $\Gamma$ be a connected graph. A* maximal tree $T$ *in $\Gamma$ is a connected subgraph of $\Gamma$ of genus 0 such that the set of vertices of $T$ coincides with the set of vertices of $\Gamma$.*

Each connected graph contains a maximal tree.

**Definition 2.4.** *An* enhanced graph *is a connected graph enhanced with the following additional data:*

- *a choice of an orientation on all edges of* $\Gamma$*;*
- *a choice of a cyclic order on the set of half-edges adjacent to every* 3*-valent vertex of* $\Gamma$*;*
- *a choice of a maximal tree* $T$ *in* $\Gamma$*;*
- *a choice of a basepoint* $p$ *in* $T$*.*

*For a given edge* $e \in E(\Gamma)$ *and a given orientation on* $e$*, a vertex* $v$ *adjacent to* $e$ *is referred to as a* source *of* $e$ *if* $e$ *is directed outwards with respect to* $v$*. If* $e$ *is directed inwards with respect to* $v$*, the vertex* $v$ *is referred to as a* sink *of* $e$*.*

We use the same notation for an enhanced graph and its underlying graph if this does not lead to an ambiguity.

The starting point of the following construction is an enhanced graph $\Gamma$ with $n$ ends and genus equal to $g$. Our construction of a presentation of $H_{g,n}$ consists of the following steps.

(1) Regard $\Gamma$ as a one-dimensional cell complex. Subdivide each inner edge of $\Gamma$ by a 2-valent vertex. Glue an additional 1-cell, in such a way that both boundary components get glued to the same point, to each of the newly added 2-valent vertices, and to each of the ends of $\Gamma$. All the newly added 1-cells are referred to as *circles*. Choose an orientation on all the circles in an arbitrary way. The obtained cell complex is homotopy equivalent to a bouquet of $|E(\Gamma)| + g$ circles. Its fundamental group is the free group of the corresponding rank. We present it in the following way.

   - *Generators corresponding to edges in* $T$ are presented by based loops that start in $p$, go along edges of $T$ to the attachment point of the circle corresponding to the edge we are interested in, make a complete revolution around it in the positive direction, and return back to $p$ along edges of $M$.
   - *Generators corresponding to edges that are not in* $T$ are presented by based loops with base $p$. According to the chosen orientation, each edge that does not belong to $T$ has a beginning and an end. Start in $p$, go along edges of $T$ to the beginning of the edge we are interested in, go along this edge to the attachment point of the circle, make a complete revolution in the positive direction, and return back the same way.

   If $e$ is an edge of $\Gamma$, the generator corresponding to $e$ described above is denoted by $p_e$.

   - *Generators corresponding to nontrivial cycles in* $\Gamma$ are presented by based loops with base $p$. The set of such generators is in a bijection with the set of edges of $\Gamma$ that do not belong to $T$. Start in $p$, go along edges of $M$ to the beginning of the edge that is not in $T$, continue along it to the end, and return back to $p$ along edges of $T$.

   If $e$ is an edge of $\Gamma$ that is not contained in the maximal tree, the generator described above is denoted by $g_e$.

(2) For each of the 3-valent vertices of $\Gamma$, prepare a 2-cell with an oriented boundary. Attach these cells to the cell complex in such a manner that the boundary goes as follows. Begin in a 3-valent vertex, go along one of the edges attached to it to the attachment point of the circle, and go along it using the following convention: if we have arrived there along the orientation of the edge, go along the circle in the positive direction. Otherwise go along it in the negative direction. Then return back to the 3-valent vertex we have started with and continue along the next half-edge according to the chosen cyclic order. Repeat the procedure for each of the edges adjacent to the vertex. Figure 1 illustrates how the gluing is performed.

   The obtained cell complex is homotopy equivalent to a connected oriented surface of genus $g$ with $n$ punctures. Each of the attached 2-cells provides a relation in its fundamental group. Note that for every edge $e$ not contained in the maximal tree, the generator $g_e$ appears in only one of the obtained relations. Van Kampen's theorem implies that these relations are the defining relations of $H_{g,n}$.

The Euler characteristic of a connected graph of genus $g$ is equal to $1-g$. This allows us to compute explicitly the number of generators and relations in the presentation in terms of the genus and the number of branching points. Therefore, we have just proved the following lemma.

**Lemma 2.5.** *The group* $H_{g,n}$ *admits a presentation with* $4g - 3 + 2n$ *generators and* $2g - 2 + n$ *relations.*

Each 2-cell used in the construction corresponds to a pair of pants. So the obtained presentation corresponds to a pair-of-pants decomposition of a surface.

**Example.** Let us consider the enhanced graph shown in Fig. 2 with the blackboard cyclic ordering of half-edges in its vertices. The maximal tree is chosen as shown. The basepoint is the only 3-valent vertex in the maximal tree.

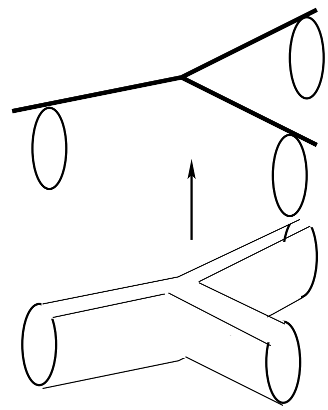

Fig. 1. An example of gluing a 2-cell to a graph with attached circles. A vicinity of a 3-valent vertex of the graph is shown in the top part of the figure.

The corresponding group presentation is as follows:

$$H_{2,1} = \langle p_a, p_b, p_c, p_d, p_e, g_d, g_e \,|\, p_a^{-1} p_b p_c = p_b^{-1} g_e^{-1} p_e g_e p_d = p_c^{-1} g_d^{-1} p_d g_d p_e = 1 \rangle.$$

## 3. Reformulation of the problem in terms of the group algebra

The problem of enumerating principal $G$-bundles is equivalent to the problem of enumerating morphisms from the fundamental group of a punctured surface to a finite group $G$ (for the details, see, for example, [9]).

The number of possible morphisms between two groups does not depend on their presentations. Thus, considering the presentation constructed in the previous section, we split the main problem of this note into two parts:

- Fix the conjugacy classes of *all* images of the generators of $H_{g,n}$ corresponding to the edges of the chosen graph $\Gamma$, and count the number of such morphisms.
- Take the sum over all possible choices of conjugacy classes of images of the inner edges.

A similar method was discussed in [1], where the authors considered the problem from the tropical point of view. In a sense, on the one hand, the main result of this paper is a generalization of the result of [1] to the case of an arbitrary finite group. On the other hand, it is nothing else but an application of a well-known argument, which can be found, for example, in [3], but in a slightly nonstandard form. That is why we present a complete proof of the main theorem.

To deal with the first part, we need some algebraic notions.

**Definition 3.1.** *Let $G$ be a finite group. By the* complex group algebra *of $G$ we mean the algebra of complex-valued functions on $G$ with the convolution as the multiplication. Its center is denoted by $Z\mathbb{C}[G]$. The set of conjugacy classes of $G$ is denoted by $\Lambda_G$.*

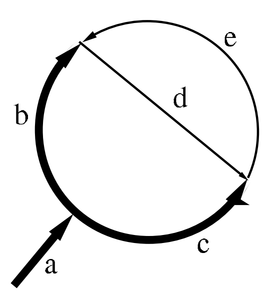


Fig. 2. An enhanced graph. The maximal tree is shown in bold. The orientation of an edge is shown by an arrowhead. The cyclic ordering of half-edges obeys the blackboard convention. The basepoint is the only 3-valent vertex of the maximal tree.

The center $Z\mathbb{C}[G]$ of the complex group algebra has a natural basis indexed by $\Lambda_G$ (see, for example, [9, Appendix A]). Namely, if $\mu$ is an arbitrary conjugacy class, then the corresponding basis element is its characteristic function (here we regard a conjugacy class as a subset of $G$). For a fixed group $G$, denote the basis element of $Z\mathbb{C}[G]$ corresponding to a conjugacy class $\mu$ by $f_\mu$. This basis is referred to as the *standard basis.* The trace function $\mathrm{tr}_G$, endowing $Z\mathbb{C}[G]$ with the Frobenius structure, is defined as follows.

**Definition 3.2.** *Let $G$ be a finite group. The* trace function *is the linear functional $\mathrm{tr}_G\colon Z\mathbb{C}[G] \to \mathbb{C}$ defined on the elements of the standard basis as follows:*

$$\mathrm{tr}_G(f_\mu) = \begin{cases} 1 & \text{if } \mu = (1), \\ 0 & \text{otherwise,} \end{cases}$$

*where $(1)$ denotes the conjugacy class of the unit element in $G$.*

We omit the subscript $G$ in the notation for the trace function when this does not cause an ambiguity.

**Definition 3.3.** *Let $\Gamma$ be a graph and $G$ be a finite group. By a* coloring of $\Gamma$ *we mean a map $c\colon E(\Gamma) \to \Lambda_G$. If $\Gamma$ is an enhanced graph, given a vertex $v \in V^0(\Gamma)$, an edge $e \in E_v$, and a coloring $c$, by $\bar{c}_v(e)$ we mean $c(e)$ if $e$ is oriented in such a way that $v$ is a source for it, and the reciprocal conjugacy class otherwise.*

The procedure from Sec. 2 allows one to construct a presentation of the fundamental group of a three times punctured sphere with 3 generators and 1 relation: one should use an enhanced graph $D$ with one 3-valent vertex and three 1-valent ones. Fix such a presentation and denote by $p_e$ the generator corresponding to an edge $e \in E(D)$. Let $G$ be an arbitrary finite group and $c$ be a coloring of $D$. Denote by $N_D(c)$ the set of morphisms $H_{0,3} \to G$ such that the image of the generator $p_e$ is contained in the conjugacy class $c(e)$ for all $e \in E(D)$.

A tautological corollary of the definition of the trace function reads as follows.

**Lemma 3.4.** *Let $G$ be a finite group, and let $c$ be a coloring of the enhanced graph $D$ described above. Then the set of morphisms from $H_{0,3}$ to $G$ such that the image of the generator $p_e$ corresponding to an edge $e \in E(D)$ is contained in $c(e)$ has the cardinality*

$$\#N_D(c) = \mathrm{tr}\Big( \prod_{e \in E(D)} f_{\bar{c}_v(e)} \Big),$$

*where $v$ denotes the only vertex of $D$.*

Now we are ready to prove the following lemma.

**Lemma 3.5.** *Let $G$ be a finite group. Let $\Gamma$ be an enhanced graph with coloring $c$. Then the set $N_\Gamma(c)$ of morphisms $H_{g,n} \to G$ such that the image of the generator $p_e$ corresponding to an edge $e \in E(\Gamma)$ belongs to the conjugacy class $c(e)$ has the cardinality*

$$\#N_\Gamma(c) = |G|^g \bigg( \prod_{v \in V^0} \mathrm{tr}\Big( \prod_{e' \in E_v} f_{\bar{c}_v(e')} \Big) \bigg) \prod_{e \in E^0} \frac{1}{\mathrm{tr}(f_{c(e)} f_{c^{-1}(e)})} ,$$

*where $c^{-1}(e)$ denotes the conjugacy class reciprocal to $c(e)$ for every edge $e \in E(\Gamma)$.*

*Proof.* First of all, note that for every conjugacy class $\mu \in \Lambda_G$, the value $\mathrm{tr}(f_\mu f_{\mu^{-1}})$ equals the number of group elements belonging to $\mu$.

Consider the enhanced graph $\Gamma'$ obtained from $\Gamma$ by cutting all the edges not belonging to the maximal tree and attaching 1-valent vertices to the remaining half-edges. The newly obtained leaves inherit the orientation from the edges and the coloring from the cut edges. The cyclic orientation in the set of half-edges adjacent to a vertex remains the same. Denote the inherited coloring by $c'$. The enhanced graph $\Gamma'$ allows us to construct a presentation of $H_{0,g+n}$.

The maximal tree for $\Gamma'$ coincides with $\Gamma'$.

The number $\#N_{\Gamma'}(c')$ is computed inductively. Pick an arbitrary vertex $v$ and make a random choice of images of the generators $p_e$ corresponding to the edges $e \in E_v$. According to Lemma 3.4, this can be done in $\mathrm{tr}\big( \prod_{e \in E_v} f_{\bar{c}(e)} \big)$ different ways.

Let $v'$ be the second vertex adjacent to $e$. Since the choice of an image for the generator $p_e$ is already made, the choice of images for the generators corresponding to the remaining edges adjacent to $v'$ can be performed in

$$\frac{\operatorname{tr}\left(\prod_{e'\in E'_v} f_{\bar{c}(e')}\right)}{\operatorname{tr}(f_{c(e)} f_{c^{-1}(e)})}$$

ways.

Any two vertices in $\Gamma'$ are connected with at most one edge. It follows that

$$\#N_{\Gamma'}(c') = \prod_{v\in V^0(\Gamma)} \operatorname{tr}\biggl(\prod_{e'\in E_v} f_{\bar{c}_v(e')}\biggr) \prod_{e\in E^0(\Gamma')} \frac{1}{\operatorname{tr}(f_{c(e)} f_{c^{-1}(e)})}.$$

There is a natural map $b\colon E(\Gamma') \to E(\Gamma)$, which sends an edge of $\Gamma'$ to the corresponding edge of $\Gamma$ before cutting. The map $b$ gives rise to a morphism $\phi\colon H_{0,g+n} \to H_{g,n}$. At the level of generators, this morphism is described in the following way:

- If $e'$ is an inner edge of $\Gamma'$, or $e'$ is a leaf shared by both $\Gamma$ and $\Gamma'$, then the generator $p_{e'}$ goes to the generator corresponding to the edge $b(e') \in E(\Gamma)$.
- If $e'$ is a leaf of $\Gamma'$ coming from a cut edge of $\Gamma$, and its end is a sink of $e'$, then the generator $p_{e'}$ goes to the generator corresponding to the edge $b(e') \in E(\Gamma)$.
- If $e'$ is a leaf of $\Gamma'$ coming from a cut edge of $\Gamma$, and its end is a source of $e'$, then the generator $p_{e'}$ goes to $g^{-1}_{b(e')} p_{b(e')} g_{b(e')}$.

The check that $\phi$ is indeed a group morphism is straightforward.

The morphism $\phi$ induces a natural map $\phi^*\colon N_\Gamma(c) \to N_{\Gamma'}(c')$.

For an edge $e \in E(\Gamma)$ that is not contained in the maximal tree, denote by $e^+$ its preimage such that its end is a sink of $e^+$. The other preimage is denoted by $e^-$.

Let $f' \in N_{\Gamma'}(c')$. We can construct from it a morphism $f \in N_\Gamma(c)$ in the following way. If $e \in \Gamma$ is contained in the maximal tree, then the image $f(p_e)$ coincides with the image $f'(p_{b^{-1}(e)})$. Otherwise the image $f(p_e)$ coincides with $f'(p_{e^+})$.

The only thing left is to fix images of the generators $g_e$ corresponding to the edges of $\Gamma$ that are not contained in the maximal tree. We fix them in such a way that $f'(p_{e^-}) = f(g_e^{-1}) f'(p_{e^+}) f(g_e)$. The verification that $f$ is indeed a group morphism is straightforward.

Moreover, it is clear that $\phi^*(f) = f'$. Since for every $e$ not belonging to the maximal tree, the choice of an image of $g_e$ can be performed in

$$\frac{|G|}{\operatorname{tr}(f_{c(e)} f_{c^{-1}(e)})}$$

ways, and the choices for different edges can be made completely independently, because each $g_e$ appears in only one relation, we arrive at the assertion of the lemma. $\square$

Recall that in fact we are interested in the number of morphisms $H_{g,n} \to G$ such that only the conjugacy classes of images of the generators corresponding to the leaves are fixed.

**Definition 3.6.** *Let $G$ be a finite group and $\Gamma$ be a graph. A* boundary condition *is a map $M\colon E(\Gamma)\backslash E^0(\Gamma) \to \Lambda_G$. The set of all possible colorings $c\colon E(\Gamma) \to \Lambda_G$ such that $M = c|_{E(\Gamma)\backslash E^0(\Gamma)}$ is denoted by $C(M)$. Abusing notation, denote $\cup_{C(M)} N_\Gamma(c) = N_\Gamma(M)$.*

**Theorem 3.7.** *Let $G$ be a finite group, $\Gamma$ be an enhanced graph, and $M$ be a boundary condition. Then the number $\#N_\Gamma(M) = \sum\limits_{c\in C(M)} \#N_\Gamma(c)$ is invariant under the following moves of enhanced graphs:*

- *another choice of a maximal tree in $\Gamma$;*
- *a change of the orientation of an edge $e \in E^0(\Gamma)$;*
- *a change of the cyclic order on the half-edges adjacent to a vertex $v \in V^0(\Gamma)$;*
- *a local transformation of $\Gamma$ as shown in Fig. 3.*

*Proof.* A different choice of a maximal tree in $\Gamma$ respects every single term in the sum $\sum\limits_{c\in C(M)} \#N_\Gamma(c)$.

Assume that an edge $e \in E^0(\Gamma)$ has coloring $c(e)$. Changing the orientation of the inner edge $e$ and switching its coloring to the reciprocal coloring $c^{-1}(e)$ also preserves every single term in the sum under consideration.

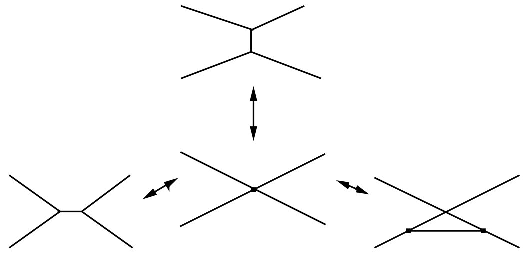

Fig. 3. The local graph moves. The figure shows all possible resolutions of a 4-valent vertex.

The independence on the choice of a cyclic ordering of the half-edges adjacent to a vertex $v \in V^0(\Gamma)$ follows from the commutativity of $Z\mathbb{C}[G]$.

The invariance under a local move of $\Gamma$ follows from the fact that the number of possible morphisms from one group to another (with some fixed conditions) does not depend on the presentation of the group. But we would like to give an alternative argument.

The commutative algebra $Z\mathbb{C}[G]$ can be endowed with a Hermitian product. Namely, consider the semilinear conjugation acting on a basis element $f_\mu$ for $\mu \in \Lambda_G$ in the following way:

$$\bar{\cdot}: \ f_\mu \mapsto f_{\mu^{-1}}.$$

The product is defined by the formula $(a, b) = \operatorname{tr}(a\bar{b})$ for $a, b \in Z\mathbb{C}[G]$. The basis vectors of $Z\mathbb{C}[G]$ form an orthogonal basis with respect to this product. Consider the trace of the product of four basis vectors of $Z\mathbb{C}[G]$ corresponding to conjugacy classes $\mu_1, \mu_2, \mu_3, \mu_4 \in \Lambda_G$. Due to the general properties of Hilbert spaces and the commutativity of $Z\mathbb{C}[G]$, for any permutation $\sigma \in S_4$ this trace can be expanded as follows:

$$\operatorname{tr}(f_{\mu_1} f_{\mu_2} f_{\mu_3} f_{\mu_4}) = \sum_{\nu \in \Lambda_G} \frac{\operatorname{tr}(f_{\mu_{\sigma(1)}} f_{\mu_{\sigma(2)}} f_\nu) \operatorname{tr}(f_{\mu_{\sigma(3)}} f_{\mu_{\sigma(4)}} f_{\nu^{-1}})}{\operatorname{tr}(f_\nu f_{\nu^{-1}})}.$$

This formula implies the desired invariance. □

Since the space of all connected 1-3-valent graphs is connected with respect to the above-mentioned moves (see, for example, [8]), the only data that matters for the computation of the number $\#N_\Gamma(M)$ is the genus of the enhanced graph $\Gamma$, the orientation of the leaves of $\Gamma$, and the boundary condition $M$. If we are dealing with a group where every conjugacy class is self-reciprocal, such as a symmetric group $S_d$, then the orientation of the leaves is also completely irrelevant.

**Example.** Let $\Gamma(2, 1)$ be a connected 1-3-valent graph of genus 2 with 1 leaf and $(-1)$ be the conjugacy class of the nonunit element of $S_2$. Endow $\Gamma$ with an arbitrary enhancement. We are interested in the computation of the number $\#N_{\Gamma(2,1)}(M)$, where $M$ is the boundary condition that assigns the class $(-1)$ to the only leaf of $\Gamma(2, 1)$. As stated above, this number is independent of the choice of an enhancement of $\Gamma$.

Let $c$ be a coloring of $\Gamma(2, 1)$. The multiplication table of $Z\mathbb{C}[S_2]$ implies that if there exists a vertex $v \in V^0(\Gamma(2, 1))$ such that there are exactly one or three edges adjacent to $v$ colored by $(-1)$, then this coloring does not contribute to the sum.

On the contrary, if every vertex $v \in V^0(\Gamma(2, 1))$ is adjacent to exactly zero or two edges with coloring $(-1)$, then this coloring contributes 1.

This actually means that $\#N_{\Gamma(2,1)}(M) = 0$.

On the other hand, for the same group $S_2$, for a graph $\Gamma(2, 2)$ of genus 2 with 2 leaves, and for the boundary condition $M'$ that assigns $(-1)$ to every leaf of the graph, the number $\#N_{\Gamma(2,2)}(M')$ is equal to 16, which coincides with the prediction of the Frobenius formula (see, for example, [9, Appendix A]).

## 4. Complex Feynman calculus

This section can be regarded as a short exposition of the theory of Feynman calculus.

Feynman calculus is a powerful tool for enumerating graphs. We will apply it to compute generating functions for the numbers of principal $G$-bundles. Briefly speaking, the machinery of Feynman calculus works as follows. Consider a graph with edges colored by a finite number of colors. Feynman calculus provides us with a rule how to assign a number to every vertex and every edge of such a graph. The weight of a graph is calculated as the product of the above-mentioned numbers over all vertices and edges divided by the order of the automorphism group of the graph. Then we take the sum of the weights over all possible graphs. It turns out that this sum can be interpreted as the result of computing an integral. For the details on Feynman calculus in the real case, we refer the reader to [5] or [9].

M. Mulase and J. Yu in [10] studied Feynman calculus over a von Neumann algebra. In fact, our considerations are a generalization of the methods of Mulase and Yu to the case of graphs with 1-valent vertices.

In a sense, our considerations can be treated as a theory of Feynman calculus over a space of diagonal matrices.

Here we will be interested in complex Feynman calculus. The reason is that complex calculus works for any finite group $G$, while real calculus works only in the cases where every conjugacy class of $G$ is a self-reciprocal class.

To proceed we will need some preliminary considerations.

**Definition 4.1.** *By $\mathbb{C}^n$ we mean the $n$-dimensional complex vector space with a chosen orthonormal basis $\{f_i\}_{i=1}^n$ with respect to a chosen Hermitian product. The measure $dm$ on $\mathbb{C}^n$ is defined as $dm = \prod_{i=1}^n d\operatorname{Re} z_i \, d\operatorname{Im} z_i$, where $z_i$ is the coordinate corresponding to the basis element $f_i$.*

**Lemma 4.2.** *Let $A$ be an endomorphism of $\mathbb{C}^n$, expressed in the standard basis as a positive definite symmetric real-valued matrix, and let $p$ be an arbitrary vector in $\mathbb{C}^n$. The following equalities hold:*

(1) $\int\limits_{\mathbb{C}^n} e^{-\langle Az,z\rangle} \, dm = \frac{\pi^n}{\det A}$;

(2) $\int\limits_{\mathbb{C}^n} e^{-\langle Az,z\rangle+\langle p,z\rangle+\langle z,p\rangle} \, dm = \frac{\pi^n}{\det A} e^{<A^{-1}p,p>}$.

*Proof.* The first equality can be deduced from the fact that every sesquilinear positive definite form can be diagonalized by the action of $U(n)$. Thus it suffices to verify this formula in the one-dimensional case, where it is nothing else but the product of two Gaussian integrals.

The second equality follows from the first one by the change of variable $z \mapsto z + A^{-1}p$. $\square$

The lemma immediately implies the equalities

$$\int\limits_{\mathbb{C}^n} z_i \bar{z}_j e^{-\langle Az,z\rangle} \, dm = \frac{\pi^n}{\det A} A^{-1}_{ij} \quad \text{for all} \quad i, j,$$

$$\int\limits_{\mathbb{C}^n} z_i z_j e^{-\langle Az,z\rangle} \, dm = 0,$$

$$\int\limits_{\mathbb{C}^n} \bar{z}_i \bar{z}_j e^{-\langle Az,z\rangle} \, dm = 0.$$

They can be obtained by differentiating the second equality of the lemma by $p$ or $\bar{p}$.

**Definition 4.3.** *Let $k$ be a positive integer and $K_1$, $K_2$ be two finite sets. By a* pairing *we mean a bijection $\sigma \colon K_1 \to K_2$. The set of all pairings is denoted by $\Pi(K_1, K_2)$.*

It is clear that $\#\Pi(K_1, K_2) = \delta_{\#K_1, \#K_2}(\#K_1)!$, where $\delta_{\cdot,\cdot}$ is the Kronecker symbol.

The next theorem is a standard fact. Its proof in the real case can be found in [5]. The proof for the complex case can be obtained by the same considerations.

**Theorem 4.4** (Wick)**.** *Let $L = \{l_1, \ldots, l_m\}$, $L' = \{l'_1, \ldots, l'_n\}$ be two collections of complex linear forms on $\mathbb{C}^n$, and let $A$ be an endomorphism of $\mathbb{C}^n$ expressed in the standard basis by a symmetric positive definite real matrix. Then*

$$\int\limits_{\mathbb{C}^n} \prod_{l \in L_1} l(z) \prod_{l' \in L'} \bar{l}'(z) e^{-\langle Az,z\rangle} \, dm = \frac{\pi^n}{\det\, A} \sum_{\sigma \in \Pi(L,L')} \prod_{l \in L} \langle A^{-1}l, \sigma(l)\rangle,$$

*and the integral converges absolutely.*

Let us again stress the fact that the only way to obtain a nonzero value is to have the same number of linear and semilinear forms under the integral.

**Definition 5.1.** *Let $\Sigma$ be an oriented surface and $f, g\colon \Sigma' \to \Sigma$ be two principal $G$-bundles with a nontrivial holonomy around a finite number of points. Then $f$ and $g$ are considered to be equivalent if there exists an automorphism $h\colon \Sigma' \to \Sigma'$ such that $f = g \circ h$. The* automorphism group $\operatorname{Aut} f$ *of a principal $G$-bundle $f\colon \Sigma' \to \Sigma$ is the group of automorphisms $h\colon \Sigma' \to \Sigma'$ such that $f = f \circ h$. The number $\frac{1}{\#(\operatorname{Aut}\ f)}$ is referred to as the* weight *of the cover.*

Consider the following enumeration problem.

**Problem.** Fix $k+2$ points $q_0, \dots, q_{r+1}$ on $\mathbb{C}P^1$ and a finite group $G$. Choose two conjugacy classes $\mu$ and $\nu$ of $G$. Find the weighted number of principal $G$-bundles over $\mathbb{C}P^1$ with a nontrivial holonomy around $q_0, \dots, q_{r+1}$ such that the holonomy around $q_0$ is in $\mu$, the holonomy around $q_{r+1}$ is in $\nu$, and the holonomy around all the other points $(q_1, \dots, q_n)$ belongs to a fixed conjugacy class $\tau$.

Denote the corresponding number by $h_\tau(\mu, \nu; r)$.

As it was mentioned in the introduction, the weighted number of principal $G$-bundles with fixed conjugacy class of holonomies can be computed as the number of morphisms from the fundamental group of an $r+2$ times punctured sphere to the group $G$ divided by the order of $G$.

**Definition 5.2.** *Let $G$ be a finite group and $\{f_\mu\}_{\mu\in\Lambda_G}$ be the standard basis of $Z\mathbb{C}[G]$. Endow $Z\mathbb{C}[G]$ with a Hermitian product $< \cdot,\cdot >_G$ such that the basis $\{f_\mu\}_{\mu\in\Lambda_G}$ is orthonormal with respect to it. Let $\tau \in \Lambda_G$ be a fixed conjugacy class. Denote by $A_\tau(\beta)$ the linear endomorphism of $Z\mathbb{C}[G]$ represented in the standard basis by the matrix with entries*

$$(A_\tau)_{\mu\nu}(\beta) = \operatorname{tr}(f_{\mu^{-1}} f_\nu (1 - \beta f_\tau))$$

*for $\mu, \nu \in \Lambda_G$.*

Note that for any $\tau$ there exists a neighborhood of the point $\beta = 0$ such that the matrix $A_\tau(\beta)$ is nondegenerate.

**Theorem 5.3.** *Let $G$ be a finite group, and let $\mu$, $\nu$, and $\tau$ be three elements of $\Lambda_G$. The generating function $h^\tau_{\mu,\nu}(\beta) = \sum_r h_\tau(\mu, \nu; r)\beta^r$ is given by the following formula:*

$$h^\tau_{\mu,\nu}(\beta) = \frac{1}{Z_\tau} \frac{\operatorname{tr}(f_{\mu^{-1}} f_\mu)\operatorname{tr}(f_{\nu^{-1}} f_\nu)}{\#G} \int\limits_{Z\mathbb{C}[G]} z_\mu \bar{z}_\nu e^{-\langle A_\tau(\beta)z, z\rangle}\, dm,$$

*where $Z_\tau$ denotes the value of the integral*

$$Z_\tau = \int\limits_{Z\mathbb{C}[G]} e^{-\langle A_\tau(\beta)z, z\rangle}\, dm.$$

*Proof.* Let $\Gamma_r$ be an enhanced graph of genus 0 with $r+2$ leaves such that every inner vertex of $\Gamma_r$ is adjacent to at least one leaf. It is clear that there are exactly two inner vertices $v_1$ and $v_2$ of $\Gamma_r$ adjacent to two leaves. Denote one of the leaves adjacent to $v_1$ by $e_1$, and one of the leaves adjacent to $v_2$ by $e_2$. Orient the leaves of $\Gamma_r$ so that for each of them the corresponding end is a sink.

Denote by $M_r$ the boundary condition that assigns the class $\mu$ to $e_1$, the class $\nu$ to $e_2$, and the class $\tau$ to every other leaf of $\Gamma_r$.

It is clear from our previous considerations that $h_\tau(\mu, \nu, r) = \frac{\# N_{\Gamma_r}(M_r)}{\#G}$.

According to Theorem 3.7, the number of covers can be computed by the formula

$$h_\tau(\mu, \nu; r) = \frac{1}{\#G} \sum_{c\in C(M_r)} \left( \prod_{v\in V_0(\Gamma_r)} \operatorname{tr}\Big( \prod_{e'\in E_v} f_{\bar{c}_v(e')} \Big) \right) \prod_{e\in E_0(\Gamma_r)} \frac{1}{\operatorname{tr}(f_{c(e)} f_{c^{-1}(e)})}.$$

Let $k$ be a nonnegative integer. Denote by $X_r$ the set of maps

$$x\colon \{1, \dots, r\} \to (\Lambda_{S_d})^2.$$

The notation $x_i(k)$ for $i = 1, 2$ stands for the corresponding component of the map. By an automorphism of a map $x \in X_r$ we mean an automorphism $h$ of the set $\{1, \dots, r\}$ such that $x = x \circ h$. The group of automorphisms of a map $x \in X_r$ is denoted by $\operatorname{Aut} x$.

Let $D$ be a linear automorphism of $Z\mathbb{C}[G]$ represented in the standard basis by a matrix $D_{\mu\nu} = \operatorname{tr}(f_{\mu^{-1}} f_\nu)$.

Let us discuss the denominator $Z_\tau$ from the statement of the theorem. Using the absolute convergence of the integral in a neighborhood of the point $\beta = 0$, we expand the expression in the following manner:

$$Z_\tau = \sum_{r=0}^{\infty} \beta^r \sum_{x \in X_r} \frac{1}{\#\mathrm{Aut}\ x} \int\limits_{Z\mathbb{C}[G]} \prod_{k=1}^{r} \mathrm{tr}\big(f_{x_1(k)} f_{x_2(k)} f_\tau\big) z_{x_1(k)} \bar{z}_{x_2(k)} e^{-\langle Dz, z\rangle}\, dm.$$

To compute this expression, we apply Theorem 4.4. For $x \in X_r$, let $\Pi^x$ denote the subset of $\Pi_r = \Pi(\{1, \dots, r\}, \{1, \dots, r\})$ such that $x_1(k) = x_2^{-1}(\sigma(k))$ for any $\sigma \in \Pi^x$. The elements of $\Pi^x$ are called *admissible pairings.* For any nonnegative integer $r$, we obtain

$$Z_\tau = \frac{\pi^{\#\Lambda_G}}{\det\, D} \sum_{r=0}^{\infty} \beta^r \sum_{x \in X_r} \sum_{\sigma \in \Pi^x} \frac{1}{\#\mathrm{Aut}\, x} \prod_{k=1}^{r} \frac{\mathrm{tr}\big(f_{x_1(k)} f_{x_2(k)} f_\tau\big)}{\mathrm{tr}\big(f_{x_1(k)} f_{x_2^{-1}(\sigma(k))}\big)}.$$

This expression can be interpreted in the following way. Let a *flower* be an elementary piece of a graph consisting of two vertices connected by an edge and two oriented half-edges attached to one of the vertices. For one of the half-edges, the adjacent vertex is a sink (we call this half-edge *left*); for the other one, the adjacent vertex is a source (this half-edge is *right*). The edge connecting the vertices of a flower is oriented in such a way that its end is a sink. Each half-edge of a flower is colored by an element of $\Lambda_G$, and the only edge of a flower is colored by the fixed class $\tau$ of $G$.

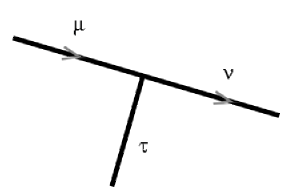


Fig. 4. A flower. The left half-edge is colored by $\mu \in \Lambda_G$, the right half-edge is colored by $\nu \in \Lambda_G$.

Every pairing $\sigma \in \Pi_r$ corresponds to an element of the symmetric group $S_r$. Namely, set the result of applying this element to $k \in \{1, \dots, r\}$ equal to $\sigma(k)$. By abuse of notation, we denote a pairing and the corresponding permutation by the same letter.

Pick an arbitrary $x \in X_r$. For each $k \in \{1, \dots, r\}$ prepare a flower with the left half-edge colored by $x_1(k)$ and the right half-edge colored by $x_2(k)$. Every admissible pairing provides a recipe to assemble a colored 1-3-valent graph. Namely, the right half-edge of the $k$th flower gets glued to the left half-edge of the $\sigma(k)$th flower. The resulting graph can be disconnected, but each of its connected components has genus 1. The number of connected components of the graph equals the number of cycles in the permutation $\sigma$. For an example, see Fig. 5.

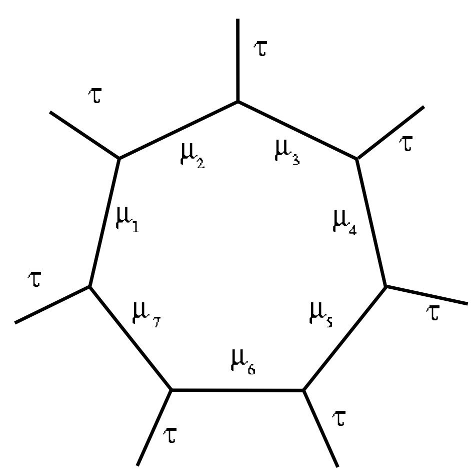


Fig. 5. An example of assembling a graph. The directions of the edges are not shown.

We claim that Theorem 3.7 implies that $Z_\tau$ is a generating function for the weighted number of principal $G$-bundles over a collection of disjoint tori such that every nontrivial holonomy around a point is in the class $\tau$ multiplied by $\pi^{\#\Lambda_G}/\det D$. Note that, on the other hand,

$$Z_\tau = \pi^{\#\Lambda_G}/\det A_\tau(\beta).$$

Now consider the integral

$$Z = \int\limits_{Z\mathbb{C}[G]} z_\mu \bar{z}_\nu e^{-\langle A_\tau(\beta)z,z\rangle}\, dm.$$

As it was mentioned during the study of $Z_\tau$, we can use the expansion

$$\sum_{r=0}^{\infty} \beta^r \sum_{x\in X_r} \frac{1}{\#\mathrm{Aut}\, x} \int\limits_{Z\mathbb{C}[G]} z_\mu \bar{z}_\nu \prod_{k=1}^{r} \mathrm{tr}\big(f_{x_1(k)} f_{x_2(k)} f_\tau\big) z_{x_1(k)} \bar{z}_{x_2(k)} e^{-\langle Dz,z\rangle}\, dm.$$

The only difference from $Z_\tau$ is in two distinguished forms $z_\mu$ and $\bar{z}_\nu$. Again applying Theorem 4.4, we see that the computation of this integral differs from the computation of $Z_\tau$ only by the existence of two additional elementary blocks. One is represented by a vertex with adjacent half-edge oriented in such a way that the vertex is a source, colored by $\mu$. The other is represented by a vertex with adjacent half-edge oriented in such a way that the vertex is a sink, colored by $\nu$. Our rules for assembling graphs from flowers and these two additional details are the same. For each graph we pick only one copy of each of these details.

Note that both distinguished details always contribute to the same connected component of a graph of genus 0. This connected component is exactly the graph we discussed at the beginning of the proof.

The division by $Z_\tau$ allows us to get rid of the contribution of all components of genus 1. The origin of the factor $\mathrm{tr}(f_\mu^{-1} f_\mu)\mathrm{tr}(f_\nu^{-1} f_\nu)/\#G$ is obvious. □

A direct application of Theorem 4.4 leads to the following corollary.

**Corollary 5.4.** *We have*

$$h^\tau_{\mu,\nu}(\beta) = \frac{\mathrm{tr}(f_{\mu^{-1}} f_\mu)\mathrm{tr}(f_{\nu^{-1}} f_\nu)}{\#G} (A_\tau^{-1}(\beta))_{\mu,\nu},$$

*where $A_\tau^{-1}(\beta)$ is the matrix inverse to $A_\tau(\beta)$.*

Since the matrix $A_\tau(\beta)$ is linear in $\beta$, we obtain the following result.

**Corollary 5.5.** *For any finite group $G$ and any $\mu,\nu,\tau \in \Lambda_G$, the generating function $h^\tau_{\mu,\nu}(\beta)$ is a rational function in $\beta$.*

Another simple theorem follows from our representation of these numbers in terms of graphs.

**Theorem 5.6.** *The generating function for the numbers $h_\tau(\mu,\nu;r)$ satisfies the following equation:*

$$h^\tau_{\mu,\nu}(\beta) + \beta\frac{\partial}{\partial\beta} h^\tau_{\mu,\nu}(\beta) = (\#G) \sum_{\lambda\in\Lambda_{S_d}} \frac{h^\tau_{\mu,\lambda}(\beta) h^\tau_{\bar{\lambda},\nu}(\beta)}{\mathrm{tr}(f_{\lambda^{-1}} f_\lambda)}.$$

This equation corresponds to the presentation of a genus 0 chain-like graph as the union of two graphs of the same type along their leaves.

## 6. Double Hurwitz numbers

Let $d$ be a positive integer, and let $\tau_d \in \Lambda_{S_d}$ be the class of a transposition. In this section, we present the matrices $A_{\tau_d}(\beta)$ and $(A_{\tau_d}(\beta))^{-1}$ for $d = 2, 3, 4$.

For $d = 2$, the basis elements are ordered in the following manner. The first one corresponds to the partition (11), and the second one corresponds to (2):

$$A_{\tau_2}(\beta) = \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix},$$

$$(A_{\tau_2}(\beta))^{-1} = \begin{pmatrix} \frac{1}{1-\beta^2} & \frac{\beta}{1-\beta^2} \\ \frac{\beta}{1-\beta^2} & \frac{1}{1-\beta^2} \end{pmatrix}.$$

For $d = 3$, we use the ordering $(111), (12), (3)$:

$$A_{\tau_3}(\beta) = \begin{pmatrix} 1 & -3\beta & 0 \\ -3\beta & 3 & -6\beta \\ 0 & -6\beta & 2 \end{pmatrix},$$

$$(A_{\tau_3}(\beta))^{-1} = \begin{pmatrix} \frac{6\beta^2-1}{9\beta^2-1} & -\frac{\beta}{9\beta^2-1} & -\frac{3\beta^2}{9\beta^2-1} \\ -\frac{\beta}{9\beta^2-1} & -\frac{1}{27\beta^2-3} & -\frac{\beta}{9\beta^2-1} \\ -\frac{3\beta^2}{9\beta^2-1} & -\frac{\beta}{9\beta^2-1} & \frac{3\beta^2-1}{9\beta^2-1} \end{pmatrix}.$$

For $d = 4$, the ordering of the basis elements is $(1111), (112), (22), (13), (4)$:

$$A_{\tau_4}(\beta) = \begin{pmatrix} 1 & -6\beta & 0 & 0 & 0 \\ -6\beta & 6 & -6\beta & -24\beta & 0 \\ 0 & -6\beta & 3 & 0 & -12\beta \\ 0 & -24\beta & 0 & 8 & -24\beta \\ 0 & 0 & -12\beta & -24\beta & 6 \end{pmatrix},$$

and $(A_{\tau_4}(\beta))^{-1}$ is equal to

$$\begin{pmatrix} \frac{24\beta^4-34\beta^2+1}{144\beta^4-40\beta^2+1} & -\frac{20\beta^3-\beta}{144\beta^4-40\beta^2+1} & \frac{24\beta^4+2\beta^2}{144\beta^4-40\beta^2+1} & -\frac{3\beta^2}{36\beta^2-1} & \frac{16\beta^3}{144\beta^4-40\beta^2+1} \\ -\frac{20\beta^3-\beta}{144\beta^4-40\beta^2+1} & -\frac{20\beta^2-1}{864\beta^4-240\beta^2+6} & \frac{12\beta^3+\beta}{432\beta^4-120\beta^2+3} & -\frac{\beta}{72\beta^2-2} & \frac{8\beta^2}{432\beta^4-120\beta^2+3} \\ \frac{24\beta^4+2\beta^2}{144\beta^4-40\beta^2+1} & \frac{12\beta^3+\beta}{432\beta^4-120\beta^2+3} & \frac{72\beta^4-30\beta^2+1}{432\beta^4-120\beta^2+3} & -\frac{3\beta^2}{36\beta^2-1} & -\frac{24\beta^3-2\beta}{432\beta^4-120\beta^2+3} \\ -\frac{3\beta^2}{36\beta^2-1} & -\frac{\beta}{72\beta^2-2} & -\frac{3\beta^2}{36\beta^2-1} & \frac{12\beta^2-1}{288\beta^2-8} & -\frac{\beta}{72\beta^2-2} \\ \frac{16\beta^3}{144\beta^4-40\beta^2+1} & \frac{8\beta^2}{432\beta^4-120\beta^2+3} & -\frac{24\beta^3-2\beta}{432\beta^4-120\beta+3} & -\frac{\beta}{72\beta^2-2} & -\frac{20\beta^2-1}{864\beta^4-240\beta^2+6} \end{pmatrix}.$$

For example,

$$h^{\tau_4}_{(4),(4)}(\beta) = \frac{6 \cdot 6}{4!} \frac{1 - 20\beta^2}{864\beta^4 - 240\beta^2 + 6} = \sum_{k=0}^{\infty} \left( \frac{36^k}{8} + \frac{4^{k-1}}{2} \right) \beta^{2k}.$$

Let us compare this generating function with a known result. The double Hurwitz numbers with $\mu = (d)$ are known as the one-part double Hurwitz numbers. Consider the following function:

$$\varsigma(z) = e^{z/2} - e^{-z/2} = 2\,\sinh(z/2).$$

Considering the particular case $\mu = \nu = (d)$, we use a formula derived in [7] and reproved in [6]. In our notation, this formula reads as

$$\sum_{r=0}^{\infty} \frac{z^r}{r!} h^{\tau_d}((d), (d); r) = \frac{1}{d^2} \frac{\varsigma(d^2 z)}{\varsigma(dz)}.$$

Substituting $d = 2, 3, 4$ and applying the inverse Borel transform

$$\frac{1}{d^2} \int_0^{\infty} e^{-t} \frac{\varsigma(d^2 zt)}{\varsigma(dzt)}\, dt,$$

we see a perfect agreement with the above-mentioned expressions.

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